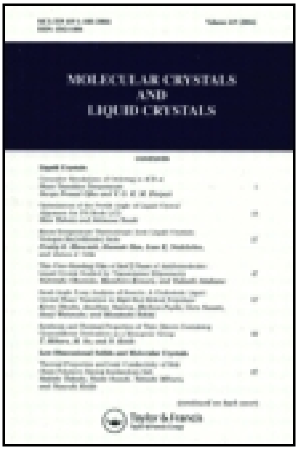

Molecular Crystals and Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl20

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H. Millar [a] & G. Mckay [a]
[a] Department of Mathematics, University of Strathclyde, Glasgow, Scotland, U.K.

Version of record first published: 31 Aug 2006

To cite this article: H. Millar & G. Mckay (2005): Director Orientation of a Twisted Nematic Under the Influence of an In-Plane Magnetic Field, Molecular Crystals and Liquid Crystals, 435:1, 277/[937]-286/[946]

To link to this article: http://dx.doi.org/10.1080/15421400590955253

Mol. Cryst. Liq. Cryst., Vol. 435, pp. 277/[937]–286/[946], 2005

ISSN: 1542-1406 print/1563-5287 online
DOI: 10.1080/15421400590955253

Director Orientation of a Twisted Nematic Under the Influence of an In-plane Magnetic Field

H. Millar
G. Mckay
Department of Mathematics, University of Strathclyde, Glasgow, Scotland, U.K.

We consider an incompressible nematic liquid crystal that exhibits an equilibrium twist profile across the cell due to in-plane but opposing anchoring directions at the plates. We examine the director orientation under the influence of a magnetic field applied in a direction parallel to the plates. Characteristic switching times are determined as the angle of twist on the plates and the direction of the applied field are varied. Furthermore, extending classical analyses we discuss Freedericksz transitions and critical field strengths.

Keywords: characteristic times; Freedericksz transition; magnetic field; twisted nematic

INTRODUCTION

The response of nematics to magnetic or electric fields (or a combination of both) can provide measurements for physical quantities such as rotational viscosity or curvature elastic constants (see, for example, Pieranski *et al.* [1]). However, Oh-E *et al.* [2] and Sun and Zhang [3] demonstrate that the behaviour of the director can be greatly affected by the presence of even a small amount of net twist across the cell or pretilt at the bounding plates. In splay or bend geometries with strong anchoring at a fixed (non-zero) pretilt on the plates, it is well-known that nematics display a thresholdless Freedericksz transition. Details of this and related analyses may be found in Stewart [4].

We examine the influence of an in-plane magnetic field on a nematic liquid crystal which is subject to a net twist across the cell.

Address correspondence to Geoffrey McKay, Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow, G1 1XH, Scotland, U.K. E-mail: gmck@maths.strath.ac.uk

This net twist is the result of strong in-plane anchoring conditions on the parallel bounding plates. We calculate equilibrium director profiles satisfying the Ericksen–Leslie equations as the field strength, net director twist and the angle of the field are allowed to vary. Furthermore, we examine dynamic switching times for the different types of profile and determine numerically the dependence of critical field strength on the fixed surface twist. Classical Freedericksz transition analyses are then extended to incorporate the effect of the net twist. The theoretical expression for the critical field strength derived from the analysis is shown to be in very good agreement with numerical predictions.

MODEL

We consider an incompressible nematic liquid crystal, infinite in the xy-plane and constrained in the z-direction by two bounding plates a distance d apart. We represent the liquid crystal via a unit vector $\boldsymbol{n}$, which we assume also lies in the xy-plane,

$$\boldsymbol{n} = (\cos\phi(z, t),\ \sin\phi(z, t),\ 0),$$

where the angle $\phi(z, t)$ denotes the twist of the director about the normal to the planes at any given time t. Initially we introduce a magnetic field, $\boldsymbol{H}$, of magnitude H applied in the y-direction,

$$\boldsymbol{H} = H(0, 1, 0). \tag{1}$$

(Later, we will consider an in-plane magnetic field parallel to the boundary plates but twisted at an angle to the y-direction.) The behaviour of the director in the bulk of the sample can be affected by the presence of a net twist across the cell. We examine this influence by introducing in-plane but opposing strong anchoring directions at the plates (see Fig. 1),

$$\phi(0, t) = -\phi_p, \quad \phi(d, t) = \phi_p, \quad \text{for all } t \geq 0.$$

We examine director profiles and switching times for the sample as the anchoring direction ϕ_p is allowed to vary.

ANALYSIS

Initially we follow the analysis of Stewart and Faulkner [5], thereby reducing the Ericksen–Leslie equations [6] for the dynamic behaviour of nematic liquid crystals (in the absence of flow) to

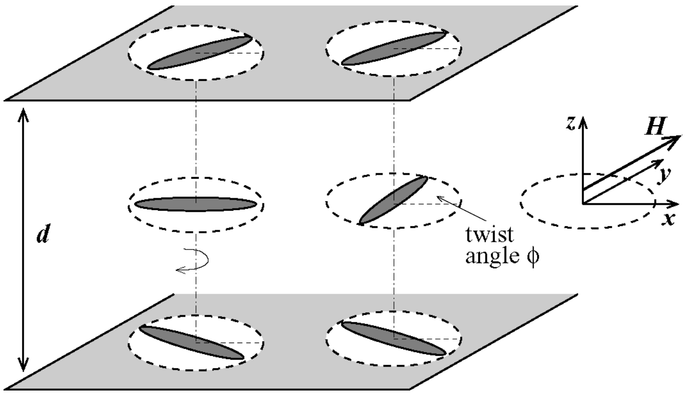

FIGURE 1 Schematic of the liquid crystal cell.

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \hat{z}^2} + \lambda \sin \phi \cos \phi, \tag{2}$$

where we have non-dimensionalized by introducing the variables

$$\tau = \frac{K_2}{\gamma_1 d^2} t, \quad \hat{z} = \frac{1}{d} z, \quad \lambda = \frac{\chi_a H^2 d^2}{K_2}.$$

Director twist $\phi \equiv \phi(\hat{z}, \tau)$, γ_1 is the (non-negative) twist viscosity coefficient, K_2 is the Frank elastic constant associated with twist and χ_a is the magnetic anisotropy. Parameter λ is a measure of the relative contribution of the elastic and magnetic field effects. We assume positive magnetic anisotropy, and hence λ is also assumed positive. The director profile satisfies Equation (2) subject to boundary conditions

$$\phi(0, \tau) = -\phi_p, \quad \phi(1, \tau) = \phi_p, \quad \phi(\hat{z}, 0) = \Phi(\hat{z}), \tag{3}$$

where $\Phi(\hat{z})$ is an (as yet unspecified) initial twist profile.

Equilibrium Profiles and Switching Times

From Equation (2), equilibrium director twist profiles, $\bar{\phi}(\hat{z})$, are solutions of the system

$$\frac{d^2 \bar{\phi}}{d\hat{z}^2} + \lambda \sin \bar{\phi} \cos \bar{\phi} = 0, \quad \bar{\phi}(0) = -\phi_p, \quad \bar{\phi}(1) = \phi_p. \tag{4}$$

However, from our numerical calculations it is clear that these profiles exhibit a bifurcation, or Freedericksz transition, even when the net twist is present across the cell. These bifurcations occur when the magnetic field strength is close to H_{cr}, the critical Freedericksz transition value for a twisted nematic and no net twist,

$$H_{cr} = \frac{\pi}{d}\sqrt{\frac{K_2}{\chi_a}}.$$

Stewart [4] presents a derivation of this classical threshold for a twisted nematic geometry. In all subsequent calculations and Figures we set the physical parameters as follows: $K_2 = 3 \times 10^{-7}$ dyne, $d = 1 \times 10^{-5}$ cm, $\chi_a = 5 \times 10^{-6}$.

When $H < H_{cr}$, only one type of solution is observed, regardless of the net twist across the cell. These solutions are monotonic and anti-symmetric with respect to the centre of the cell. However, Figure 2 demonstrates that more than one equilibrium profile may exist when $H \gtrsim H_{cr}$. The secondary profiles are non-monotonic and asymmetric. (Here *non-monotonic* is regarded as a profile for which there exists $\hat{z}_m \in [0, 1]$ such that $\phi'(\hat{z}_m) = 0$ and $\bar{\phi}(\hat{z}_m)$ is a maximum. The case where $\hat{z}_m = 1$, i.e. $\phi'(1) = 0$, corresponds to the critical bifurcation field strength and is discussed later.) The magnetic field contribution to the free energy of our system is minimized when $\phi = 90°$. Therefore, as the field strength is increased through a critical value close to H_{cr}, the symmetry imposed by the strong anchoring is broken as the director attempts to minimize the magnetic energy (by increasing twist in the bulk of the cell). From our numerical calculations, we

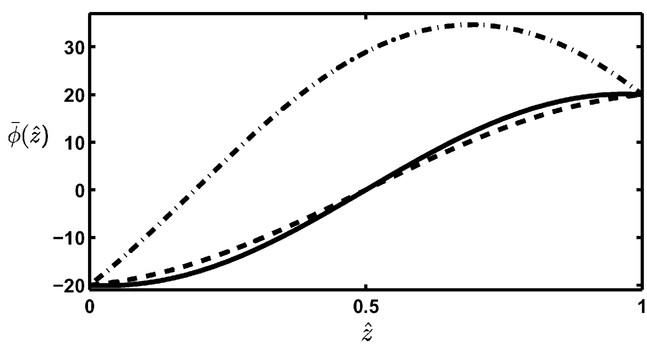

FIGURE 2 Equilibrium profiles $\bar{\phi}(\hat{z})$ (in degrees) for $\phi_p = 20°$ as magnetic field strength is varied. (– – – –): anti-symmetric pre-bifurcation profile, $H = 0.9H_{cr}$, (——) monotonic post-bifurcation profile, $H = 1.1H_{cr}$, (–·–·–): non-monotonicpost-bifurcation profile, $H = 1.1H_{cr}$.

derive that the critical field strength, H_{bifur}, at which the bifurcation occurs is related to the surface twist ϕ_p via

$$H_{\text{bifur}}(\phi_p) \approx H_{\text{cr}}(1 + 0.25\phi_p^2). \tag{5}$$

SWITCHING TIMES

We may also examine characteristic switch-on times associated with the equilibrium profiles. Assume that at time $\tau = 0$ the magnetic field is suddenly switched on to strength H, i.e. $\lambda > 0$ for $\tau > 0$. Prior to this time the twist exhibits an equilibrium linear twist through angle $2\phi_p$. Therefore, we consider system (2), (3) with an initial profile $\Phi(\hat{z}) = (2\hat{z} - 1)\phi_p$. Linearizing about a static equilibrium profile $\bar{\phi}(\hat{z})$ corresponding to field strength H, the linearized system for (small) perturbation $\eta(\hat{z}, \tau) = \phi(\hat{z}, \tau) - \bar{\phi}(\hat{z})$ may be written as

$$\frac{\partial \eta}{\partial \tau} = \frac{\partial^2 \eta}{\partial \hat{z}^2} + \eta\lambda\cos(2\bar{\phi}), \quad \eta(0, \tau) = \eta(1, \tau) = 0, \quad \eta(\hat{z}, 0) = \Phi(\hat{z}) - \bar{\phi}(\hat{z}). \tag{6}$$

Perturbation $\eta(\hat{z}, \tau)$ is now approximated in suitable basis functions

$$\eta(\hat{z}, \tau) = \sum_{j=1}^{N} a_j(\tau)\sin(j\pi\hat{z}),$$

for some finite N. Employing a straightforward spectral analysis, we may rewrite (6) in matrix form with $\boldsymbol{a} = (a_1(\tau), \ldots, a_N(\tau))^{\mathrm{T}}$,

$$\frac{d\boldsymbol{a}}{d\tau} = \mathcal{D}\boldsymbol{a}, \quad \text{where } [\mathcal{D}]_{jk} = 2\int_0^1 (\lambda\cos(2\bar{\phi}) - k^2\pi^2)\sin(j\pi\hat{z})\sin(k\pi\hat{z})\,d\hat{z}. \tag{7}$$

Provided H is close to H_{cr}, the eigenvalues of $\mathcal{D}$ are negative, therefore $\eta(\hat{z}, \tau) \to 0$ as $\tau \to \infty$, i.e. the director approaches its equilibrium configuration as τ increases. Subsequently, a characteristic switch-on time associated with profile $\bar{\phi}(\hat{z})$ may be defined as

$$t_{\text{on}} = \frac{\gamma_1 d^2}{K_2\omega},$$

where $-\omega(<0)$ is the eigenvalue of $\mathcal{D}$ with smallest magnitude. Note, this definition is in terms of time t (seconds), not non-dimensional variable τ.

In Figure 3 we plot the reciprocal of ω (which is directly proportional to the switch-on time) corresponding to both the anti-symmetric and

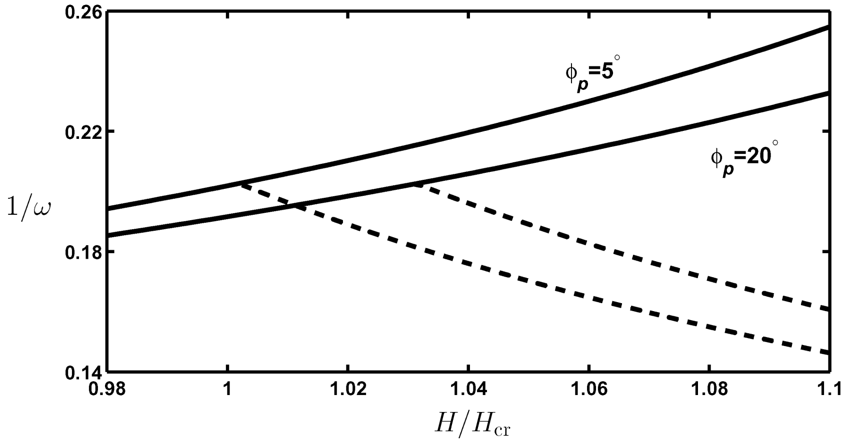

FIGURE 3 Reciprocal of eigenvalue ω, proportional to switching times, for $\phi_p = 5^\circ,\ 20^\circ$. (———): anti-symmetric profiles, (– – –): non-monotonic, post-bifurcation profiles.

post-bifurcation profiles as the field strength varies. Not only is the post-bifurcation profile more energetically favourable than its anti-symmetric counterpart (details omitted), but its characteristic times are also smaller (despite a greater change in the profile from its linear state). Also, while characteristic times for the anti-symmetric solutions are larger when ϕ_p is small, the opposite is true for the post-transition profiles.

Variable Magnetic Field Direction

We now examine the effect of twisting the magnetic field in our system such that it still lies parallel to the plates, but at angle α to the x-direction, i.e.

$$\boldsymbol{H} = H(\cos\alpha,\ \sin\alpha,\ 0).$$

Therefore our analysis in the previous Sections corresponds to the special case $\alpha = 90^\circ$. The non-dimensional governing equation is now

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial^2\phi}{\partial\hat{z}^2} - \lambda\sin(\phi-\alpha)\cos(\phi-\alpha), \tag{8}$$

and most of the subsequent analysis is analogous to that examined previously.

However, in this case we find that only one equilibrium solution exists for a given magnetic field, except when α is an integer multiple

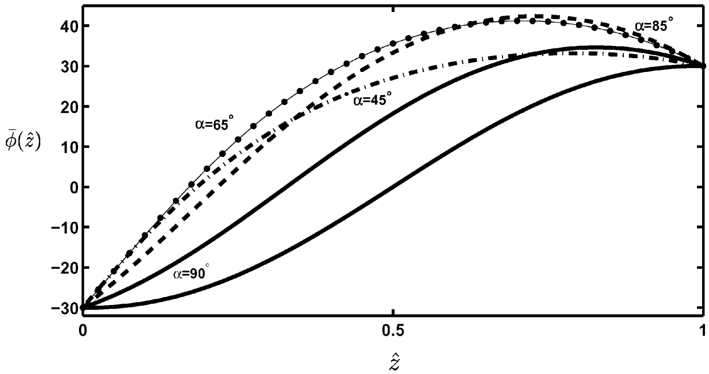

FIGURE 4 Equilibrium profiles $\bar{\phi}(\hat{z})$ (in degrees) when $H = 1.1\,H_{\rm cr}$ and field direction α is varied. (−·−·−): $\alpha = 45°$, (●–●–●): $\alpha = 65°$, (− − −): $\alpha = 85°$, (———): anti-symmetric and non-monotonic profiles for $\alpha = 90°$.

of 90°, as illustrated in Figure 4. We can no longer obtain the anti-symmetric profile due to the presence of the (symmetry-breaking) field direction α. In Figure 5 we examine the characteristic times as α is varied. The only bifurcation occurs (for the range of α considered) when $\alpha = 90°$, as discussed previously. Furthermore, characteristic times are decreased as the angle α decreases from 90°.

FREEDERICKSZ TRANSITION

We shall now return to the equilibrium equation and boundary conditions defined by Equations (4) for the case $\alpha = 90°$. In particular, we examine critical conditions for a Freedericksz transition as the field strength is increased. Note that since the boundary conditions are in opposing directions there is no constant solution satisfying this system. Multiplying the second order differential equation in $\bar{\phi}$ by $d\bar{\phi}/d\hat{z}$ and integrating produces the first integral

$$\frac{\partial\bar{\phi}}{\partial\hat{z}} = \pm\sqrt{c^2 - \lambda\sin^2\bar{\phi}}, \quad \text{for some constant } c. \tag{9}$$

For the anti-symmetric equilibrium profile discussed previously, the twist angle may be assumed to increase monotonically from $-\phi_p$ to ϕ_p. Therefore, we shall adopt the plus sign in (9) while considering these profiles (the minus sign case is treated analogously). Integrating (9) over the height of the cell, we obtain

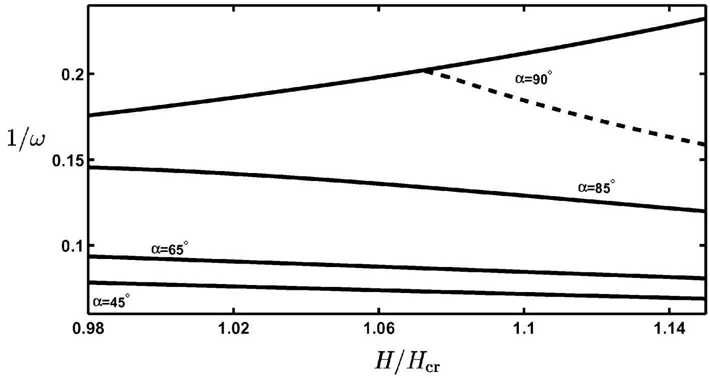

FIGURE 5 Reciprocal of eigenvalue ω when $\phi_p = 30^\circ$ and field direction is varied, $\alpha = 45^\circ$, 65°, 85°, 90° (including bifurcation at 90°).

$$\int_{-\phi_p}^{\phi_p} \frac{1}{\sqrt{c^2 - \lambda sin^2 \tilde{\phi}}} d\tilde{\phi} = 1. \tag{10}$$

Integral equation (10) allows us to calculate c for a given value of λ. For example, when $H = 0$, i.e. $\lambda = 0$, we find that $c = 2\phi_p$. From (8) we can then derive the initial profile considered previously, $\Phi(\hat{z}) = (2\hat{z} - 1)\phi_p$.

For $H > 0$, a full set of monotonic, anti-symmetric equilibrium twist profiles different from $\Phi(\hat{z})$ may be obtained via

$$\hat{z} = \int_{-\phi_p}^{\bar{\phi}} \frac{1}{\sqrt{c^2 - \lambda \sin^2 \tilde{\phi}}} d\tilde{\phi}, \tag{11}$$

where c is determined for each value of H from (10). As solutions to (11) exist for all values of H, we may deduce that monotonic equilibrium profiles exhibit a thresholdless Freedericksz transition, in agreement with our numerical calculations.

If we now consider non-monotonic equilibrium profiles, we cannot assume that the twist angle is always increasing; therefore we have to consider the plus and minus signs in (9) separately. We have already introduced $\hat{z}_m \in [0, 1]$ to represent the height at which the twist obtains its maximum. Let $\phi_m (\geq \phi_p)$ denote this maximum value, i.e.

$$\bar{\phi}(\hat{z}_m) = \phi_m, \quad \frac{d\bar{\phi}}{d\hat{z}}(\hat{z}_m) = 0. \tag{12}$$

We may now adopt the plus sign in (9) for $0 \le \hat{z} \le \hat{z}_m$ and the negative sign for $\hat{z}_m < \hat{z} \le 1$. From (9) it follows that $c^2 = \lambda \sin^2 \phi_m$, and the following integrals are obtained

$$\sqrt{\lambda}\hat{z} = \int_{-\phi_p}^{\bar{\phi}} \frac{1}{\sqrt{\sin^2 \phi_m - \sin^2 \tilde{\phi}}} d\tilde{\phi}, \qquad 0 \le \tilde{z} \le \tilde{z}_m, \tag{13}$$

$$\sqrt{\lambda}(1 - \hat{z}) = -\int_{\bar{\phi}}^{\bar{\phi}_p} \frac{1}{\sqrt{\sin^2 \phi_m - \sin^2 \tilde{\phi}}} d\tilde{\phi}, \qquad \hat{z}_m \le \tilde{z} \le 1. \tag{14}$$

Equations (13) and (14) together furnish analytical expressions for the non-monotonic, symmetry-breaking director twist profiles for a given surface twist angle. However, in order for (13), (14) to provide meaningful descriptions for a given choice of parameters, we require formulae for the maximum twist ϕ_m and corresponding height $\hat{z}_m$.

Now consider the case when $0 < \hat{z}_m < 1$. Since $\phi_m = \bar{\phi}(\hat{z}_m)$ is a maximum, there exists $\hat{z}_p \in (0, \hat{z}_m)$ such that $\bar{\phi}(\hat{z}_p) = \bar{\phi}(1) = \phi_p$. From (13) it follows that

$$\sqrt{\lambda}\hat{z}_p = \int_{-\phi_p}^{\phi_p} \frac{1}{\sqrt{\sin^2 \phi_m - \sin^2 \tilde{\phi}}} d\tilde{\phi}. \tag{15}$$

By combining (15) with (13) and (14), we can show that $1 - \hat{z}_p = 2(1 - \hat{z}_m)$. We can also employ differential equation (4) and a Taylor series to prove that $\bar{\phi}(\hat{z})$ is symmetric about $\hat{z}_m$. Re-arranging the Taylor series we derive an expression for $\hat{z}_m$,

$$\hat{z}_m \approx 1 - \left(\frac{4(\phi_m - \phi_p)}{\lambda \sin(2\phi_m)} \right)^{1/2}. \tag{16}$$

Equation (16) provides a relationship between the surface twist angle, the maximum twist attained by the director, field strength and the physical parameters. The maximum twist in the cell, ϕ_m, may now be approximated by substituting the expression for $\hat{z}_m$ into (13).

Finally, the transition from anti-symmetric, monotonic to symmetry-breaking, non-monotonic profiles is characterized by a critical field strength such that $\bar{\phi}'(1) = 0$ and $\phi_m = \phi_p$, i.e. $z_m = 1$. In this case, (13) may be written as the sum of an elliptic integral of the first kind and a complete elliptic integral of the first kind (details omitted). By taking the limit as $\phi_m \to \phi_p$ and rearranging, we obtain analytically

$H_{\text{bifur}}(\phi_p)$, the critical Freedericksz transition field strength for given twist ϕ_p,

$$H_{\text{bifur}}(\phi_p) = \frac{2}{\pi} H_{\text{cr}} K(\sin\phi_p) \approx H_{\text{cr}}\left(1 + \frac{1}{4}\phi_p^2\right),$$

where $K(\sin\phi_p)$ is the complete elliptic integral of the first kind of modulus $\sin\phi_p$, which in turn we have expanded for small ϕ_p via a Maclaurin series. This expression is in excellent agreement with the critical field dependence (5) obtained from our numerical calculations.

SUMMARY

We have examined the Ericksen–Leslie equations for a twisted nematic sample subject to an in-plane magnetic field and a net twist across the cell (in the absence of flow). For a fixed net twist, there is a critical field strength above which there exists a symmetry-breaking, non-monotonic equilibrium director profile. Compared to the monotonic, anti-symmetric solutions, the post-bifurcation profile is associated with a lower energy and smaller characteristic switch-on times. However, changing the direction of the in-plane field can have a major effect on the system. By adapting a classical Freedericksz transition analysis, we have also derived analytical expressions for the post-transitional director twist and the dependence of critical field strength on the net twist.

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